Non-Equilibrium Beta Processes in Neutron Stars: A Relationship between the Net Reaction Rate and the Total Emissivity of Neutrinos

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ABSTRACT

Several different processes could be changing the density in the core of a neutron star, leading to a departure from β equilibrium, quantified by the chemical potential difference $\delta\mu \equiv \mu_n - \mu_p - \mu_e$. The evolution of this quantity is coupled to that of the star's interior temperature T by two functions that quantify the rate at which neutrino-emitting reactions proceed: the net reaction rate (difference between β decay and capture rates), $\Gamma_{\rm net}(T, \delta\mu)$, and the total emissivity (total energy emission rate in the form of neutrinos and antineutrinos), $\epsilon_{\rm tot}(T, \delta\mu)$. Here, we present a simple and general relationship between these variables, $\partial\epsilon_{\rm tot}/\partial\delta\mu = 3\Gamma_{\rm net}$, and show that it holds even in the case of superfluid nucleons. This relation may simplify the numerical calculation of these quantities, including superfluid reduction factors.

Subject headings: dense matter — neutrinos — stars: neutron

1. INTRODUCTION

The simplest weak interaction process that could proceed in the core of a neutron star is the so-called direct Urca (or *Durca*) process, which consists of the two successive reactions, β decay and capture.

$$n \to p + e + \bar{\nu}_e \qquad\qquad p + e \to n + \nu_e. \tag{1}$$

It is the most powerful of the neutrino processes potentially leading to the cooling of the neutron star. However, these reactions are allowed only if each of the Fermi momenta of

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neutrons (n), protons (p), and electrons (e) is smaller than the sum of the two others (triangle condition). This implies a (highly uncertain) threshold in the matter density for the direct Urca processes. Another reaction, which overcomes this restriction, is the so-called modified Urca (or *Murca*) process, which involves an additional spectator nucleon:

$$n + N \rightarrow p + e + \bar{\nu}_e + N \qquad\qquad p + e + N \rightarrow n + \nu_e + N, \tag{2}$$

where the additional nucleon N can be either a neutron or a proton. The matter is transparent to neutrinos, which escape freely, transporting their energy away from the star.

Furthermore, these two reactions bring nucleons into the state of beta (or chemical) equilibrium, which determines the concentration of neutrons and protons. The beta equilibrium condition $\mu_n = \mu_p + \mu_e$ involving the chemical potentials of the constituent particles implies the equality of $\bar{\nu}_e$ and ν_e emission rates (or the net reaction rate set equal to zero). If a slight departure from equilibrium $\delta\mu = \mu_n - \mu_p - \mu_e$ is produced by any external or macroscopic phenomenon that changes the density of a fluid element, one of the two reactions in equations (1) or (2) becomes more intense and changes the fraction of protons and neutrons towards the new equilibrium values (Le Châtelier's principle). In the subsequent evolution, the chemical imbalance affects the stellar interior temperature T by increasing the phase space available to the products of the neutrino-emitting reactions and by converting chemical energy into thermal energy, and the temperature affects the chemical imbalance by also determining the rate at which reactions proceed. Several authors have investigated external processes that induce non-equilibrium β reactions, such as radial pulsation (Finzi 1965; Finzi & Wolf 1968), gravitational collapse (Haensel 1992; Gourgoulhon & Haensel 1993), a changing rotation rate (Reisenegger 1995, 1997; Fernández & Reisenegger 2005; Reisenegger et al. 2006), and a hypothetical time-variation of the gravitational constant (Jofré et al. 2006). However, none of these has considered an adequate model for the effects of the likely Cooper pairing (superfluidity) of nucleons (see Reisenegger 1997 for a very rough estimate of their likely importance).

Two basic rates are relevant in order to follow the coupled evolution of T and $\delta\mu$ (see, e. g., Reisenegger 1995; Fernández & Reisenegger 2005): the total (neutrino and antineutrino) emissivity (energy per unit volume per unit time), $\epsilon_{\rm tot} = \epsilon_{\bar{\nu}} + \epsilon_{\nu}$, and the net reaction rate (number of reactions or emitted lepton number per unit volume per unit time), $\Gamma_{\rm net} = \Gamma_{\bar{\nu}} - \Gamma_{\nu}$. Recently, Villain & Haensel (2005) analyzed the effect of nucleon superfluidity on the net reaction rates, $\Gamma_{\rm net}$, calculating superfluid reduction factors for the direct and modified Urca processes by means of sophisticated numerical methods. Nevertheless, they did not evaluate the total emissivity of neutrinos, $\epsilon_{\rm tot}$, which is also required in order to compute the eventual time-evolution of T and $\delta\mu$.

In this work, we present a curious relationship between $\epsilon_{\rm tot}$ and $\Gamma_{\rm net}$, both considered

as functions of the chemical imbalance parameter $\delta\mu$, that holds even in the superfluid case:

$$\frac{\partial \epsilon_{\rm tot}}{\partial \delta\mu} = 3\Gamma_{\rm net}. \tag{3}$$

The advantage of this relation is that, having computed one of the functions numerically (as done by Villain & Haensel 2005), the other is obtained very easily, saving time in the calculation.

In the non-superfluid case, Reisenegger (1995) calculated $\epsilon_{\rm tot}$ and $\Gamma_{\rm net}$ analytically. For Durca processes,

$$\epsilon_{\rm tot}^{\rm Durca}(T,\delta\mu) = \epsilon_0^D \left(1 + \frac{1071u^2 + 315u^4 + 21u^6}{457}\right), \tag{4}$$

and

$$\Gamma_{\rm net}^{\rm Durca}(T,\delta\mu)\delta\mu = \epsilon_0^D \frac{714u^2 + 420u^4 + 42u^6}{457}, \tag{5}$$

where $u \equiv \delta\mu/(\pi T)$ is a dimensionless parameter, and $\epsilon_0^D \equiv \epsilon_{\rm tot}^{\rm Durca}(T,0) \propto T^6$ is the equilibrium emissivity. For Murca processes,

$$\epsilon_{\rm tot}^{\rm Murca}(T,\delta\mu) = \epsilon_0^M \left(1 + \frac{22020u^2 + 5670u^4 + 420u^6 + 9u^8}{11513}\right), \tag{6}$$

and

$$\Gamma_{\rm net}^{\rm Murca}(T,\delta\mu)\delta\mu = \epsilon_0^M \frac{14680u^2 + 7560u^4 + 840u^6 + 24u^8}{11513}, \tag{7}$$

with $\epsilon_0^M \equiv \epsilon_{\rm tot}^{\rm Murca}(T,0) \propto T^8$ (see Haensel 1992 for precise estimates). In these two pairs of polynomial expressions, it is easy to verify our proposed relation (eq. 3). The purpose of this paper is to show that it is valid beyond these simple cases, encompassing, for example, the cases of superfluid neutrons and/or protons. A brief and clear discussion of superfluidity in neutron star was given by Yakovlev (2001), and a rigorous and deep analysis can be found in Lombardo & Schulze (2001).

2. DERIVATION

We now show that equation (3) holds regardless of the nucleons being superfluid or not. Along this paper we will use natural units, with $\hbar = k_B = c = 1$.

We express the phase space factors as

$$d\mathbf{p_i} = p_i^2 \mathrm{d}p_i \, \mathrm{d}\Omega_i = D(E_i)\mathrm{d}E_i \, \mathrm{d}\Omega_i, \tag{8}$$

where $E_i(p_i)$ is the particle energy,[3] and

$$D(E_i) = p_i^2 \frac{dp_i}{dE_i}, \tag{9}$$

is the density of states and $\mathrm{d}\Omega_i$ is the solid angle element in the direction of $\mathbf{p_i}$. In order to show the generality of our derivation, we left the electron and nucleon densities of states[4] expressed implicitly in the following calculation. For neutrinos, we will need the explicit expression

$$d\mathbf{p}_\nu = E_\nu^2 \, \mathrm{d}E_\nu d\Omega_\nu, \tag{10}$$

and we may assume neutrino isotropy, so

$$d\mathbf{p}_\nu = 4\pi E_\nu^2 \, \mathrm{d}E_\nu. \tag{11}$$

In the interior of the neutron star, the temperature is much smaller than the Fermi temperature. On the other hand, the neutrino momentum is proportional to the temperature of the star, and the other momenta are essentially their respective Fermi momenta (degenerate matter). Therefore, we assume that $|\mathbf{p}_\nu| \ll |\mathbf{p_i}|$ $(i = n,\, p,\, e)$, and neglect $\mathbf{p}_\nu$ in the Dirac delta function of momentum. This approach will allow us to integrate more easily over the orientation of the neutrino momentum.

Using the following dimensionless variables:

$$x_i \equiv \frac{E_i - \mu_i}{T} \quad (i = n,\, p,\, e), \quad x_\nu \equiv \frac{E_\nu}{T} \quad \text{and} \quad u \equiv \frac{\delta\mu}{T}, \tag{12}$$

the total emissivity ϵ_{tot} can be expressed as

$$\epsilon_{\mathrm{tot}}(T, \delta\mu) = \frac{T^6}{(2\pi)^8} \langle |M|^2 \rangle \, \widehat{\Omega} \, \widehat{I}_{-} \tag{13}$$

(Yakovlev et al. 2001, eq. 115), where $\langle |M|^2 \rangle$ is the squared transition amplitude for the β decay and capture processes, averaged over initial and summed over final spin states; furthermore averaged over the direction of the neutrino momentum, this results in a roughly

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[3]The dispersion relation for the superfluid nucleons can be written as $E(p) = \mu + \mathrm{sgn}(p - p_F)\sqrt{v_F^2(p-p_F)^2 + \Delta^2}$, where p_F denotes the Fermi momenta. The Cooper pairing occurring around the Fermi surfaces, allows us to approximate $(1/2m)(p^2 - p_F^2) \simeq v_F^2(p-p_F)^2$, where $v_F = p_F/m^*$ and m^* are the Fermi velocity and effective particle mass for superfluid nucleons, respectively.

[4]The density of states for superfluid nucleons is $D(E) = \frac{p_F\, m^* |E-\mu|}{\sqrt{(E-\mu)^2 - \Delta^2}}\, \Theta(|E-\mu| - \Delta)$.

constant value, which can be taken out of the integrals (for more details see Shapiro & Teukolsky 1983 or Yakovlev et al. 2001):

$$\langle |M|^2 \rangle \simeq 2G_F \cos^2\theta_C (1+3g_A^2), \tag{14}$$

where G_F, θ_C, and g_A are the Fermi weak interaction constant, the Cabibbo angle ($\sin\theta_C = 0.231$), and the Gamow-Teller axial vector coupling constant, respectively. The operator $\widehat{\Omega}$ contains the integrals over the orientations of the particle momenta

$$\widehat{\Omega} \equiv 4\pi \iiint \mathrm{d}\Omega_n \mathrm{d}\Omega_p \mathrm{d}\Omega_e \, \delta(\mathbf{p_n} - \mathbf{p_p} - \mathbf{p_e}), \tag{15}$$

and $\widehat{I}_-$ integral includes the integrations over dimensionless particle energies:

$$\widehat{I}_- \equiv \int_{-\infty}^{\infty} \mathrm{d}x_e D_e f_e \int_{-\infty}^{\infty} \mathrm{d}x_n D_n f_n \int_{-\infty}^{\infty} \mathrm{d}x_p D_p f_p \int_0^{\infty} \mathrm{d}x_\nu x_\nu^3$$
$$\times \left[\delta(x_n + x_p + x_e - x_\nu + u) - \delta(x_n + x_p + x_e - x_\nu - u) \right], \tag{16}$$

where the f_i's (for $i = p, n, e$) are the Fermi-Dirac distributions, $f_i = (e_i^x + 1)^{-1}$, for some of whose arguments the signs have been redefined when using the symmetry property $f_i(x) = 1 - f_i(-x)$. The functions $D(x_i)$ are approximately symmetric around $x = 0$ for all relevant cases.

When considering situations with neutron or proton superfluidity, the possible anisotropy of gaps (appearing in the density of states function) in the expression $\widehat{I}_-$ should be taken into account in the preceding $\widehat{\Omega}$ integration.

Similarly, the net reaction rate is

$$\Gamma_{\mathrm{net}}(T, \delta\mu) = \frac{T^5}{(2\pi)^8} \langle |M|^2 \rangle \, \widehat{\Omega} \, \widehat{I}_+, \tag{17}$$

where

$$\widehat{I}_+ \equiv \int_{-\infty}^{\infty} \mathrm{d}x_e D(x_e) f_e \int_{-\infty}^{\infty} \mathrm{d}x_n D(x_n) f_n \int_{-\infty}^{\infty} \mathrm{d}x_p D(x_p) f_p \int_0^{\infty} \mathrm{d}x_\nu x_\nu^2$$
$$\times \left[\delta(x_n + x_p + x_e - x_\nu + u) + \delta(x_n + x_p + x_e - x_\nu - u) \right]. \tag{18}$$

Note that the integral now contains only two powers of x_ν.

The total emissivity (given by eq. [13]) depends on the chemical imbalance $\delta\mu$ only through the dimensionless non-equilibrium parameter u contained in $\widehat{I}_-$, in the argument of the Dirac delta function. In order to calculate the derivative $\partial\epsilon_{\mathrm{tot}}/\partial\delta\mu$, we define

$$z_\pm \equiv x_n + x_p + x_e - x_\nu \pm u, \tag{19}$$

so we can rewrite the derivative

$$\frac{\partial}{\partial u}\left[\delta(z_+)-\delta(z_-)\right]=\frac{\partial}{\partial x_\nu}\left[-\delta(z_+)-\delta(z_-)\right] \tag{20}$$

and do an integral by parts in order to obtain the derivative of equation (16)[5]:

$$\frac{\partial \widehat{I}_-}{\partial u}=\int_{-\infty}^{\infty}\mathrm{d}x_e D(x_e)f_e\int_{-\infty}^{\infty}\mathrm{d}x_n D(x_n)f_n\int_{-\infty}^{\infty}\mathrm{d}x_p D(x_p)f_p\int_0^{\infty}\mathrm{d}x_\nu(3x_\nu^2)\left[\delta(z_+)+\delta(z_-)\right]=3\widehat{I}_+. \tag{21}$$

Thus, in a straightforward way, we reach the proposed relation (eq. 3):

$$\frac{\partial\epsilon_{\mathrm{tot}}}{\partial\delta\mu}=3\Gamma_{\mathrm{net}}.$$

It is straightforward to verify that an analogous derivation can be made for modified Urca processes.

3. COOLING VS. HEATING

As a useful application of the proposed relation, we analyze the balance between heating and cooling due to β processes.

The net local heating rate can be written as

$$h_{\mathrm{net}}=\Gamma_{\mathrm{net}}\delta\mu-\epsilon_{\mathrm{tot}}, \tag{22}$$

where the first term is the rate of dissipation of chemical energy, and the second is the energy loss rate due to neutrino emission. Using our relation (eq. 3), which links ϵ_{tot} and Γ_{net}, we obtain

$$h_{\mathrm{net}}=\frac{\delta\mu^4}{3}\frac{\partial}{\partial\delta\mu}\left(\frac{\epsilon_{\mathrm{tot}}}{\delta\mu^3}\right). \tag{23}$$

If ϵ_{tot} increases faster than $\delta\mu^3$, the net heating is positive ($h_{\mathrm{net}}>0$), and viceversa.

The net heating rate can also be written as

$$h_{\mathrm{net}}=\frac{\epsilon_{\mathrm{tot}}}{3}\left(\frac{\partial\ln\epsilon_{\mathrm{tot}}}{\partial\ln\delta\mu}-3\right). \tag{24}$$

Without Cooper pairing, and in the limiting case of $\delta\mu\gg T$, we know that $\epsilon_{\mathrm{tot}}\propto\delta\mu^6$ for Durca processes, and $\epsilon_{\mathrm{tot}}\propto\delta\mu^8$ for Murca processes. For these cases, we easily reobtain that

[5]The boundary term is of the form $x^3/(1+e^x)|_0^\infty$ and vanishes in both limits.

the total energy released, $\Gamma_{\rm net}\delta\mu$, is distributed in fixed fractions among internal heating, $h_{\rm net}$, and neutrino emission, as $h_{\rm net} = \epsilon_{\rm tot} = \frac{1}{2}\Gamma_{\rm net}\delta\mu$ for Durca, and $h_{\rm net} = \frac{5}{8}\Gamma_{\rm net}\delta\mu$, $\epsilon_{\rm tot} = \frac{3}{8}\Gamma_{\rm net}\delta\mu$ for Murca processes (Fernández & Reisenegger 2005).

4. CONCLUSIONS

We have proven a simple, general relationship (eq. 3) between the main rates characterizing non-equilibrium β processes in both superfluid and non-superfluid neutron star matter. This relation could simplify the evaluation of these quantities in superfluid neutron star models, complementing numerical calculations such as those of Villain & Haensel (2005).

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